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Procedure for Optimization of Truss Space Structures with Fixed Minimum Vibration Frequency

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This paper deals with the problem of a cantilever truss space structure optimization under a frequency constraint. The considered structure, with an attached rigid body at the external free end, is typical of satellite appendages. The same value of the lowest vibration frequency of the reference truss structure, with uniform thickness over all the component beams, has been prescribed to prevent possible resonances between satellite and appendages. A minimum value of the thickness over each component beam of the analyzed truss has been established a priori. Focusing attention on the dynamic behavior, a numerical procedure (based on the classical Rayleigh–Ritz method but combined with the finite element method) has been used. Polynomial power series expansions have been used both for the dynamic variables (displacements and rotations) and for the thickness axial distribution over each component beam of the truss structure. The classical Euler–Lagrange optimality criterion has been used to find the requested solution of each optimization operation, which consists of the search for stationary conditions of the Lagrangian functional. A series of repeated optimization operations have been performed to arrive at the thickness optimized profile within each structural component, with the preestablished minimum value of the beams' thickness. It has also been possible to evaluate the weight reduction.

Nomenclature

$g^{(n)}_{i_a i_b i_c}$	=	global describing functions coefficients of the generic variable S_n
I_b	=	identification number of the generic beam
i_{bay}	=	identification number of a generic bay
k_{ij}, $k_{ij,i_{\mathrm{et}}}$	=	stiffness matrix elements and stiffness matrix elements components
$\bar{L}_{I_b}$	=	nondimensional length of a generic beam
L_t	=	sum of the dimensionless lengths of the component beams considered for the optimization operations
L_0	=	reference length
$l^{(I_b)}_{n_{ie}}$	=	local describing functions coefficients of the generic variable S_n in the I_bth beam
l_1, l_1^*, l_2	=	longitudinal lines containing aligned beams
m_{ij}, $m_{ij,i_{\mathrm{et}}}$	=	mass matrix elements and mass matrix elements components
N	=	whole number of unknown variables
N_a, N_b, N_c	=	number of dynamic variables global describing functions along the axes X, Y, and Z, respectively
N_{bays}	=	number of the truss structure component bays
N_{beams}	=	number of truss structure component beams considered for the dynamic analysis
N_{el}	=	number of local describing functions of the dynamic variables in each beam
N_{et}	=	number of coefficients of all the beams thickness series expansions
N_{opt}	=	number of all the beams considered in optimization process
N_{qt}	=	number of coefficients and describing functions of all the independent dynamic variables series expansions
N_t	=	number of coefficients and describing functions of the nondimensional thickness series expansion in each beam
q_i, q_j	=	generic Lagrangian degrees of freedom
$R^{(I_b)}_{ij}$	=	rotation matrix element connecting the axis X_i with the axis X_{lj} in the I_bth beam
S_n	=	generic independent variable
T	=	dimensionless relative component of the thickness
T_{qm}	=	mean square value of the nondimensional thickness
t	=	thickness value
t_{i_t}	=	generic coefficient of the nondimensional thickness behavior series expansion (Lagrangian variable)
t_0	=	reference truss uniform thickness value
U, V, W; U_1, U_2, U_3	=	nondimensional displacements along the x, y, z axes, respectively
U_l, V_l, W_l; U_{l1}, U_{l2}, U_{l3}	=	nondimensional displacements along the x_l, y_l, z_l axes, respectively
u, v, w; u_1, u_2, u_3	=	displacements along the x, y, z axes, respectively
u_l, v_l, w_l; u_{l1}, u_{l2}, u_{l3}	=	displacements along the x_l, y_l, z_l axes, respectively
X, Y, Z; X_1, X_2, X_3	=	dimensionless main reference system
X_l, Y_l, Z_l; X_{l1}, X_{l2}, X_{l3}	=	nondimensional local reference system
X_u	=	nondimensional longitudinal line coordinate
x, y, z; x_1, x_2, x_3	=	main reference system
x_l, y_l, z_l; x_{l1}, x_{l2}, x_{l3}	=	local reference system
α	=	azimuth angle of the oriented beam
β	=	elevation angle of the oriented beam
δ_{ij}	=	Kronecker's delta
θ_X, $-\theta_Z$, θ_Y; θ_{l1}, θ_{l2}, θ_{l3}	=	rotations along the coordinates of the local reference system
θ_x, θ_y, θ_z; θ_1, θ_2, θ_3	=	rotations along the coordinates of the main reference system
λ_i	=	generic Lagrange multiplier
ω	=	angular frequency

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ω_d	=	dimensionless angular frequency parameter
$\omega_{\min}$	=	reference shell lowest frequency

Subscripts

I_b	=	referring to the I_bth beam
i, j	=	referring to the independent dynamic variables series expansions
i_a, i_b, i_c; j_a, j_b, j_c	=	global describing functions coefficients
i_e, n_{i_e}	=	referring to the local describing functions of the dynamic variables
i_{et}, i_t	=	referring to the thickness behavior series expansion

Superscripts

(I_b)	=	referring to I_bth beam
$\bar{()}$	=	nondimensional form of ( )
(n)	=	referring to the generic independent dynamic variable

Special Symbols

C_i	=	generic governing equation
$I^{(t^2)}$	=	mean square thickness expression
$\mathcal{L}_a$	=	extended Lagrangian functional expression
$\mathcal{L}_t$	=	set of all the beams considered in the optimization operations
$\mathcal{L}_2()$	=	space of the square summable functions in ( )
$\mathcal{S}$	=	whole set of the beams considered for the dynamic analysis
$\mathcal{W}$	=	truss structure dimensionless relative weight
$\mathcal{W}_{i_{\text{et}}}$	=	truss structure relative weight component in the thickness series expansion

I. Introduction

WEIGHT minimization of aerospace vehicles structural components is one of the main goals for aircraft and spacecraft designers. But in structural optimization operations, the frequency variation range has to be limited for stability reasons of both component elements and whole spacecraft or aircraft system. In fact, the main problem arises from possible resonances between natural modes of the vibrating component elements and the whole oscillating system, thus reaching an unstable condition. In most cases, the frequencies of the whole oscillating aerospace vehicle are lower than the ones of the vibrating internal elements and appendages, not yet optimized. Consequently, the most dangerous vibration mode of each structural component is the one with the lowest frequency, for which a lower bound constraint is necessary to guarantee a safety margin.

Structural optimization of vibrating elements under frequency constraints has been a subject of widespread interest for several authors. Research work of Sippel and Warner [1] was devoted to the investigation of minimum mass design of structural elements and simple structures of sandwich type subjected to a constraint on the lowest natural frequency of vibration. Woo [2] presented an efficient structural optimization methodology for the design of minimum weight space frames under multiple natural frequency constraints. Sadek [3] set up a technique for the determination of the least weight design which satisfies the desired frequency and frequency response constraints, with upper and lower bounds on the design variables. Lee and Lim [4] proposed a simplified procedure for the optimal design with the random structural system and used a second-order perturbation method to arrive at stress, displacement, and eigenvalue constraints. Kim and Kim [5] formulated an efficient mode-tracking method for maximizing the eigenfrequencies of the vibration modes. Tong and Liu [6] presented an optimization procedure for minimum weight of truss structures under constraints of stresses, natural frequencies, and frequency responses.

Kanno and Ohsaki [7] considered an optimization problem of minimizing the structural volume under the lowest eigenvalue constraints, where both matrices of the generalized eigenvalue problem are symmetric and linear functions of the design variables. By using the Lagrange multipliers, they arrived at the Lagrangian functional expression and found the necessary and sufficient conditions for a solution of the optimization problem with multiple lowest eigenvalues by imposing stationary conditions of this functional. Sedaghati et al. [8] built a structural optimization algorithm with truss and beam-type members under single- or multiple-frequency constraints.

In this paper, a typical satellite truss appendage is considered. This is assumed to be formed by a repetitive sequence of constituting bays, for which the structural components are unidimensional tube-type beam elements, having small thickness compared to the circular cross section radius, as in the author's previous paper [9]. The structural beam model of each component is simplified according to the Mindlin–Reissner theory [10,11]; the same model was cited in a book by Timoshenko and Gere [12]. A shear stress distribution along the resultant shear force direction with maximum value at the neutral axis, which diminishes moving toward the extreme points, where it vanishes, is obtained. This beam approach is a little more sophisticated than the simplest Mindlin model [10,11] with uniform shear stress behavior.

All the component beams are assembled into a truss structure and perfectly connected at the contact points between adjacent elements. For this reason, not only the displacements but also the rotations are univocally defined at these common end points for all the adjacent beams.

A particular way to solve the problem has been devised. It is supposed that the lowest frequency of this reference truss appendage with a rigid body at the external free end is high enough to guarantee stability from coupling with the whole satellite oscillations. The problem at hand consists of searching the best thickness profile over each truss component, minimizing the total structural mass while satisfying the prescribed constraint condition on the lowest natural frequency. Further, a preestablished thickness minimum value in each optimized beam is chosen.

A semi-analytical approach, previously used by this author [9,13,14], based on the classical Rayleigh–Ritz method [15,16] and partially combined with the finite element method (FEM) [17,18], is herein developed to optimize truss space structures.

This procedure uses both global and local describing functions for the dynamic independent variables. The presence of the local functions discretizes the overall structure (like in FEM) because each one is defined only within a single component element; however, the continuity of the independent dynamic variables is guaranteed by imposing that these vanish at the boundary points between adjacent elements and for the presence of the global describing functions.

Undoubtedly, from a mathematical point of view, this procedure is very similar to a sophisticated technique of FEM, commonly called p-convergence elements method [19], considering that both utilize polynomial power series expansions with a indefinitely increasing degree of the local functions in each component element of the structure. However, there is a characteristic which distinguishes this method from all FEM numerical approaches: in this procedure, instead of grid points, we have global describing functions for the dynamic independent variables, defined in all the space containing the structure.

This procedure was previously used only for the dynamic analysis of the same truss structure, but without a rigid body at the external free end [9], whereas it has been herein employed for the optimization of the same structure. Undeniable advantages can be evinced with respect to the currently applied FEM, considering that the FEM codes require a very large number of Lagrangian degrees of freedom (DOF) and more CPU time to obtain accurate results, very close to those of the proposed computational models [9,13,14].

It has been possible to find the best thickness profile throughout all the structural component beams of the cantilever truss by using the traditional Euler–Lagrange optimality criterion [20], which is a powerful and efficient tool to find solutions of complex optimization problems.

All the aforementioned authors treated similar subjects; some of them used the same Euler–Lagrange optimization criterion, via FEM or other procedures, as in the model of Sippel and Warner [1], Kanno and Ohsaki [7], Sedaghati et al. [8], Cinquini et al. [21], and Abolbashari [22], following the same optimization operations. However, none of them used the herein developed procedure, which has been employed for the first time for application to optimization of spacecraft truss appendage structures.

In most cases, FEM codes have been used by the other authors for the dynamic analysis and optimization. Further, in their approaching models, when truss structures are optimized, there is no variation of the beam or bar thickness vs the axial coordinate of each single component of the structure. Furthermore, in most of the other authors papers, when a truss structure is analyzed, the component elements are considered pin-ended bars or beams, with hinged ends; in the present analysis, there is continuity of all the dynamic variables (displacements and rotations) at the contact points between adjacent beams.

II. Rotation Relations

Consider an oriented component beam of the truss structure, as shown in Fig. 1. Einstein's repeated indices rule will be used in all the forthcoming relations. The rotation relations between the local beam reference system x_l, y_l, z_l, with axis z_l parallel to the x, y plane, and the main reference system x, y, z, previously introduced and sufficiently illustrated [9], are briefly recalled. These can be written as

$$x_i = x_{0i} + R_{ij}^{(I_b)} x_{lj} \qquad i = 1, 2, 3 \qquad j = 1, 2, 3 \tag{1a}$$

where

$$x_1 = x \qquad x_2 = y \qquad x_3 = z \qquad x_{01} = x_0 \qquad x_{02} = y_0$$
$$x_{03} = z_0 \qquad x_{l1} = x_l \qquad x_{l2} = y_l \qquad x_{l3} = z_l \tag{1b}$$

and the rotation matrix elements $R_{ij}^{(I_b)}$ depend on the azimuth and elevation angles α and β, respectively, of the single considered I_bth beam, identified by the number I_b.

A nondimensional reference system X, Y, Z, where $X = x/L_0$, $Y = y/L_0$, and $Z = z/L_0$, and L_0 is a reference length, is introduced, along with a nondimensional X_l, Y_l, Z_l local reference system, where likewise $X_l = x_l/L_0$, $Y_l = y_l/L_0$, and $Z_l = z_l/L_0$. Obviously, the two nondimensional reference systems X, Y, Z and X_l, Y_l, Z_l are connected by the same relations as in Eq. (1a). The length L_{I_b} of the I_bth generic beam, can also be reformulated in nondimensional form $\bar{L}_{I_b} = L_{I_b}/L_0$.

The displacements u_l, v_l, w_l along the x_l, y_l, z_l axes, respectively, can be written in terms of the corresponding ones u, v, w, along the x, y, z axes, by similar relations to the ones in Eq. (1a):

$$u_{li} = R_{ji}^{(I_b)} u_j \qquad i = 1, 2, 3 \qquad j = 1, 2, 3 \tag{2a}$$

where

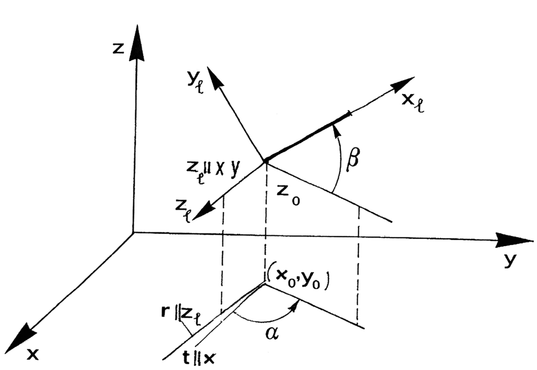

Fig. 1 Oriented beam in the main reference system.

$$u_{l1} = u_l \qquad u_{l2} = v_l \qquad u_{l3} = w_l \qquad u_1 = u \qquad u_2 = v$$
$$u_3 = w \tag{2b}$$

The nondimensional displacements $U = u/L_0$, $V = v/L_0$, and $W = w/L_0$ can be introduced, along with the corresponding ones along the axes of the local reference system $U_l = u_l/L_0$, $V_l = v_l/L_0$, and $W_l = w_l/L_0$. Obviously, the displacements U_l, V_l, W_l are related to U, V, W by the same expressions as in Eq. (2a).

Then, the rotations θ_X, $-\theta_Z$, θ_Y around the x_l, y_l, z_l axes, respectively, are introduced, which are related to the rotations θ_x, θ_y, θ_z around the x, y, z axes, via the same expressions as in Eq. (2a):

$$\theta_{li} = R_{ji}^{(I_b)} \theta_j \qquad i = 1, 2, 3 \qquad j = 1, 2, 3 \tag{3a}$$

where

$$\theta_{l1} = \theta_X \qquad \theta_{l2} = -\theta_Z \qquad \theta_{l3} = \theta_Y \qquad \theta_1 = \theta_x$$
$$\theta_2 = \theta_y \qquad \theta_3 = \theta_z \tag{3b}$$

(Do not confuse θ_X, θ_Y, θ_Z, having capital letters X, Y, Z as subscripts, with θ_x, θ_y, θ_z, having small letters x, y, z as subscripts).

III. Mathematical Model

A representation in form of polynomial series expansions, global and local, was chosen for the generic independent dynamic variable S_n in the I_bth beam, corresponding to U, V, W, θ_x, θ_y, θ_z for $n = 1, 2, \ldots, 6$, respectively, as [9]

$$S_n^{(I_b)} = g_{i_a i_b i_c}^{(n)} X^{i_a} Y^{i_b} Z^{i_c} + \sum_{i_e} l_{n_{ie}}^{(I_b)} X_l^{i_e} (\bar{L}_{I_b} - X_l) \tag{4a}$$

$$i_a = 1, 2, \ldots, N_a \qquad i_b = 0, 1, \ldots, N_b - 1$$
$$i_c = 0, 1, \ldots, N_c - 1 \qquad i_e = 1, 2, \ldots, N_{\mathrm{el}} \tag{4b}$$

because the truss structure is supposed clamped at $X = 0$.

Further, we have

$$n_{ie} = (n - 1) N_{\mathrm{el}} + i_e \qquad I_b = 1, 2, \ldots, N_{\mathrm{beams}} \tag{4c}$$

where N_{el} is the number of local describing functions of every single beam, and the identification number I_b of the generic I_bth beam can be, at most, equal to the whole number N_{beams} of the truss component beams considered in the dynamic analysis.

The global describing functions of the first series expansion in Eq. (4a) are defined in the three-dimensional X, Y, Z space containing the whole truss structure, whereas the local describing functions vs the axial coordinate X_l of the second series expansion are defined within every single component I_bth beam.

The dynamic variables series expansions coefficients, $g_{i_a i_b i_c}^{(n)}$ and $l_{n_{ie}}^{(I_b)}$ in Eq. (4a) are Lagrangian DOF, for which the total number is

$$N_{\mathrm{qt}} = N^* + 6 N_{\mathrm{beams}} N_{\mathrm{el}} \tag{5}$$

where $N^* = 6 N_a N_b N_c$. The generic dynamic Lagrangian DOF q_i is $g_{i_a i_b i_c}^{(n)}$ or $l_{n_{ie}}^{(I_b)}$, according to whether $1 \le i \le N^*$ or $N^* < i \le N_{\mathrm{qt}}$, respectively.

These describing functions, which satisfy only the geometric boundary conditions, as in the Ritz method [14,15], belong to the space $\mathcal{L}_2(\mathcal{S})$ of the square summable functions in $\mathcal{S}$ ($\mathcal{S}$ in this case is the whole set of the considered beams for the dynamic analysis). This set of polynomials form a complete system of functions satisfying the prescribed boundary conditions for the Weierstrass theorem [23]. Theoretically with these polynomial functions, it could be possible to simulate every generic, continuous and regular, modal shape.

The thickness $t(P)$ expression in a generic point P of the truss structure can be introduced:

$$t(P) = t_0 T(P) \tag{6}$$

where t_0 is the uniform thickness value of the reference structure, and T is the nondimensional relative component, with functional dependence on the position of the point P. Considering that, different from the dynamic variables behavior, there is possibility of discontinuity of the thickness value at contact points between adjacent beams, only local describing functions in each beam are used for this relative component profile. Thence, for T, the following series expansion is chosen:

$$T(P) = t_{i_t}^{(I_b)} X_l^{i_t}(P) \qquad i_t = 0, 1, 2, \ldots, N_t - 1 \qquad I_b = 1, 2, \ldots, N_{\text{opt}} \tag{7}$$

where the point P lies in the generic I_bth beam. In Eq. (7), N_{opt} is the number of the component structural beams to be optimized.

The polynomial describing functions in Eq. (7) belong to the same square summable functions space $\mathcal{L}_2(\mathcal{L}_t)$ in $\mathcal{L}_t$, which is the whole set of N_{opt} beams considered for the optimization, as the ones in Eq. (4a) in $\mathcal{S}$; however, there are not any particular boundary conditions to be satisfied at each of the beam ends by these last functions. The thickness series expansion coefficients $t_{i_t}^{(I_b)}$ in Eq. (7) are other Lagrangian DOF, for which the number is

$$N_{\text{et}} = N_t N_{\text{opt}} \tag{8}$$

It is possible to evaluate the stiffness matrix elements, taking into account that, because the thickness t is very small with respect to the tube radius R, the corresponding strain energy expression depends almost linearly on this thickness. Consequently, we can write for all stiffness matrix elements

$$k_{ij} = \sum_{I_b} \sum_{i_t} t_{i_t}^{(I_b)} k_{ij,i_{\text{et}}} \qquad I_b = 1, 2, \ldots, N_{\text{opt}} \qquad i_t = 0, 1, 2, \ldots, N_t - 1 \tag{9a}$$

$$i_{\text{et}} = (I_b - 1)N_t + i_t + 1 \qquad i_{\text{et}} = 1, 2, \ldots, N_{\text{et}} \qquad N_{\text{et}} = N_{\text{opt}} N_t \tag{9b}$$

where $k_{ij,i_{\text{et}}}$ is the contribution given by the generic component element $X_l^{i_t}$ of the series expansion in Eq. (7) in the I_bth beam, which has been evaluated in a separate paper [24].

The mass matrix elements m_{ij}, like the kinetic energy, also depend linearly on the thickness, which for Eq. (7) can be written as

$$m_{ij} = \sum_{I_b} \sum_{i_t} t_{i_t}^{(I_b)} m_{ij,i_{\text{et}}} \qquad i_{\text{et}} = (I_b - 1)N_t + i_t + 1 \qquad i_{\text{et}} = 1, 2, \ldots, N_{\text{et}} \tag{10}$$

because every component function $X_l^{i_t}$ of the thickness series expansion in the I_bth beam, gives a separate contribution $m_{ij,i_{\text{et}}}$ to the elements of this matrix, likewise determined in the same separate paper [24].

The ith governing equation of the truss structure dynamics is

$$C_i = (k_{ij} - \omega_{\min}^2 m_{ij}) q_j = 0 \qquad i, j = 1, 2, \ldots, N_{\text{qt}} \tag{11}$$

where the frequency is fixed at the minimum value $\omega_{\min}$ of the vibrating reference truss structure, having uniform thickness and an oscillating rigid body at the external free end. In other words, to keep the safety margin for the dynamic stability, the optimization operations on the truss structure must not affect its lowest vibration frequency value.

Equation (11) corresponds to the classical generalized eigenvalue problem, if the thickness profile is known in each beam and the frequencies must be determined; the solution of this eigenvalue problem has been necessary in the case of uniform thickness to evaluate the fixed minimum frequency $\omega_{\min}$. This solution has been determined by the same algorithms of the Natural Algorithm Group, Inc., (NAG) utility package [25], used to solve the generalized eigenvalue problem, in the previous dynamic analysis of the same structure, but without a rigid body at the free end [9]. These are denoted by F07FDF, which computes the Cholesky factorization; F08SEF reduces a generalized eigenproblem $Az = \lambda Bz$ to the standard form $Cy = \lambda y$; F02FCF computes the selected eigensolutions. It has been possible to obtain an accuracy of the obtained solution to 10^{-5} of the nondimensional frequencies.

In Eq. (11), the number N_{qt} of Lagrangian DOF, which characterize the dynamic state, has been defined in Eq. (5).

An input control parameter, chosen as the mean square value T_{qm} of the thickness profile, is necessary to avoid divergence and instability of the numerical solutions and, consequently, unrealistic thickness values. Thus, we have another governing equation:

$$C_{N_{\text{qt}}+1} = T_{\text{qm}} - I^{(t^2)} = 0 \tag{12}$$

where

$$I^{(t^2)} = \frac{\sum_{I_b} \int_{\bar{L}_{I_b}} T^2 \, \mathrm{d}X_l}{L_t} \qquad I_b = 1, 2, \ldots, N_{\text{opt}} \tag{13}$$

and

$$L_t = \sum_{I_b} \bar{L}_{I_b}$$

is the sum of the dimensionless lengths of all the structural beams considered in the optimization process.

The relative weight $\mathcal{W}$ (the ratio between the optimized truss structure weight and the reference structure weight with constant thickness t_0) can be written as

$$\mathcal{W} = \frac{\sum_{I_b} \int_{\bar{L}_{I_b}} T \, \mathrm{d}X_l}{L_t} \qquad I_b = 1, 2, \ldots, N_{\text{opt}} \tag{14}$$

If the thickness series expansion in Eq. (7) is taken into account, it can also be written as

$$\mathcal{W} = \sum_{I_b} \sum_{i_t} t_{i_t}^{(I_b)} \mathcal{W}_{i_{\text{et}}} \qquad I_b = 1, 2, \ldots, N_{\text{opt}} \qquad i_t = 0, 1, 2, \ldots, N_t - 1 \tag{15a}$$

$$i_{\text{et}} = (I_b - 1)N_t + i_t + 1 \qquad i_{\text{et}} = 1, 2, \ldots, N_{\text{et}} \tag{15b}$$

where

$$\mathcal{W}_{i_{\text{et}}}(I_B, i_t) = \frac{\int_{\bar{L}_{I_b}} X_l^{i_t} \, \mathrm{d}X_l}{L_t} \tag{16}$$

is the contribution to the relative weight given by the component element $X_l^{i_t}$ of the thickness series expansion in the generic I_bth beam.

Thus, an extended Lagrangian functional $\mathcal{L}_a(\mathcal{S})$ can be introduced:

$$\mathcal{L}_a(\mathcal{S}) = \mathcal{W} + \lambda_i C_i + \lambda_{N_{\text{qt}}+1} C_{N_{\text{qt}}+1} \qquad i = 1, 2, \ldots, N_{\text{qt}} \tag{17}$$

The Euler–Lagrange method consists of the search of the stationarity condition of this Lagrangian functional. Thence, we arrive at a nonlinear algebraic equations system, for which the solution is determined by the algorithm C05NBF of the NAG utility package [25], which applies Powell's hybrid method properly modified [26].

Imposing stationarity conditions of this functional, with respect to the Lagrange multipliers λ_i, leads to the dynamic governing Eq. (11). The same conditions, derived by differential operations with respect to the dynamic Lagrangian DOF q_j, give the following adjoint equations:

$$\frac{\partial \mathcal{L}_a}{\partial q_j} = \lambda_i (k_{ij} - \omega_{\min}^2 m_{ij}) = 0 \qquad i, j = 1, 2, \ldots, N_{\text{qt}} \tag{18}$$

Because the matrices with elements k_{ij} and m_{ij} are symmetric, Eq. (18) gives the same dynamic state relations in Eq. (11), for which the Lagrange multipliers λ_i can be chosen proportional to the unknown dynamic Lagrangian DOF q_i:

$$\lambda_i = \lambda_p q_i \qquad i = 1, 2, \ldots, N_{\mathrm{qt}} \tag{19}$$

where λ_p is a parameter freely chosen. Thus, the number of unknown variables of the requested solution and the equations to be satisfied can be considerably reduced.

At last, if the extended Lagrangian functional in Eq. (17), is differentiated with respect to the series expansion coefficients $t_{i_t}^{(I_b)}$ of the relative thickness, as in Eq. (7), imposing its stationary conditions, in view of Eqs. (11), (12), and (15a), leads to the following relations:

$$C_{N_{\mathrm{qt}}+1+i_{\mathrm{et}}} = \frac{\partial \mathcal{L}_a}{\partial t_{i_t}^{(I_b)}} = \mathcal{W}_{i_{\mathrm{et}}} + \lambda_i \left(\frac{\partial k_{ij}}{\partial t_{i_t}^{(I_b)}} - \omega_{\min}^2 \frac{\partial m_{ij}}{\partial t_{i_t}^{(I_b)}} \right) q_j + \lambda_{N_{\mathrm{qt}}+1} \frac{\partial I^{(t^2)}}{\partial t_{i_t}^{(I_b)}} = 0 \tag{20a}$$

$$i_{\mathrm{et}} = 1, 2, \ldots, N_{\mathrm{et}} \qquad I_b = 1, 2, \ldots, N_{\mathrm{opt}}$$
$$i_t = 0, 1, 2, \ldots, N_t - 1 \qquad i, j = 1, 2, \ldots, N_{\mathrm{qt}} \tag{20b}$$

where, if Eq. (9a) is considered, it is true that

$$\frac{\partial k_{ij}}{\partial t_{i_t}^{(I_b)}} = k_{ij, i_{\mathrm{et}}} \tag{21}$$

and in view of Eq. (10), it is also true that

$$\frac{\partial m_{ij}}{\partial t_{i_t}^{(I_b)}} = m_{ij, i_{\mathrm{et}}} \tag{22}$$

Equations (20) are the fundamental relations of the Euler–Lagrange optimality criterion.

The foregoing terms $\mathcal{W}_{i_{\mathrm{et}}}$ and the first derivative of $I^{(t^2)}$ with respect to $t_{i_t}^{(I_b)}$ in Eq. (20a) have been determined the same aforementioned paper [24].

Finally, we have $N_{\mathrm{qt}} + 1 + N_{\mathrm{et}}$ nonlinear algebraic equations:

$$C_i = \frac{\partial \mathcal{L}_a}{\partial \lambda_i} = 0 \qquad i = 1, 2, \ldots, N_{\mathrm{qt}} \qquad \text{governing equations}$$
$$C_{N_{\mathrm{qt}}+1} = \frac{\partial \mathcal{L}_a}{\partial \lambda_{N_{\mathrm{qt}}+1}} = 0 \qquad \text{thickness mean square relation}$$
$$C_{N_{\mathrm{qt}}+1+i_{\mathrm{et}}} = \frac{\partial \mathcal{L}_a}{\partial t_{i_t}^{(I_b)}} = 0 \qquad i_{\mathrm{et}} = (I_b - 1)N_t + i_t + 1$$
$$i_{\mathrm{et}} = 1, 2, \ldots, N_{\mathrm{et}} \qquad \text{weight stationary conditions relations} \tag{23}$$

and $N = N_{\mathrm{qt}} + 1 + N_{\mathrm{et}}$ unknown variables:

$$q_i \qquad i = 1, 2, \ldots, N_{\mathrm{qt}}$$
$$\text{dynamic variables series expansions coefficients}$$
$$q_{N_{\mathrm{qt}}+1} = \lambda_{N_{\mathrm{qt}}+1} \qquad \text{Lagrange multiplier} \qquad q_{N_{\mathrm{qt}}+1+i_{\mathrm{et}}} = t_{i_t}^{(I_b)}$$
$$i_t = 0, 1, \ldots, N_t - 1 \qquad I_b = 1, 2, \ldots, N_{\mathrm{opt}}$$
$$i_{\mathrm{et}} = (I_b - 1)N_t + i_t + 1 \qquad i_{\mathrm{et}} = 1, 2, \ldots, N_{\mathrm{et}}$$
$$\text{thickness series expansions coefficients} \tag{24}$$

An appropriate algorithm of the NAG utility package is used to find a solution of the nonlinear algebraic equations system, as in Eqs. (23), if input values of the unknown variables are properly chosen to start the iterative process of the computing algorithm. In fact, if these values are quite far from the corresponding ones of the true solution, big convergence problems arise for the NAG solver subroutine. A particular technique is then used to overcome these limits and find an approximate solution.

We start from the uniform thickness condition $T = 1$ throughout the truss structure, which corresponds to the input parameter value $T_{\mathrm{qm}} = 1$ of the mean square thickness. The starting values of the independent variables are obtained from the solution of the generalized eigenvalue problem in Eq. (11), which gives the requested lowest frequency $\omega_{\min}$. Then, this input parameter is lowered gradually and a series of repeated optimization operations is carried out, where the unknown variables results of each operation are the input value for the subsequent one. Also, the minimum of the thickness throughout all the component beams of the truss structure lowers gradually until it reaches the established value (which is herein 80% of the uniform thickness value t_0 of the reference structure, corresponding to $T = 0.8$).

It is obvious that the optimization procedure changes the higher frequencies and corresponding modal shapes. However, because the minimum allowed thickness value is limited to 80% of the corresponding one of the reference truss structure, we can suppose that they do not decrease so much to become close to the lowest frequency and, furthermore, with the possibility of coupling with the satellite oscillations.

This is the requested solution for the thickness profile in all the structural beams.

IV. Applications

The truss structure, in particular, the case with three component bays, and its cross section, if the thickness value is equal to t_0 ($T = 1$), are the same as in the author's previous paper [9], as shown in Fig. 2. The same number of component bays of the truss structure is adopted for the optimization operations. All the numbers useful to characterize the optimization process are as follows: $N = 178, 193, 208$; $N_{\mathrm{beams}} = 18$; $N_a, N_b, N_c = 3, 1, 3$; $N_{\mathrm{el}} = 1$; $N_{\mathrm{qt}} = 162$, $N_{\mathrm{opt}} = 15$, $N_t = 1, 2, 3$; $N_{\mathrm{et}} = 15, 30, 45$. These values are true since $N = N_{\mathrm{qt}} + 1 + N_{\mathrm{et}}$ from Eq. (23) and $N_{\mathrm{et}} = N_{\mathrm{opt}} N_t$ from Eq. (7).

As in the author's previous paper [9], where the interest of the study was limited to the dynamic analysis, the presence of the three crossbeams at the left clamped edge ($X = 0$) can be neglected, considering that these are perfectly attached to the rigid central body of the satellite.

The three crossbeams on the external free edge can be considered attached to a hypothetical rigid oscillating body; for this reason, these beams are constrained and it is impossible for these to get deformed. It is the same as if their elastic rigidity has increased indefinitely. Further, the presence of this rigid body at the free end gives rise to a concentrated inertial force. It is possible to take into account this force presence by very much increasing the distributed mass density on these beams with respect to the same of the other beams (10 times).

Consequently, these three terminal crossbeams simulate the presence of the rigid oscillating mass and are considered nonstructural inertial elements supported at the right free end of the cantilever truss structure. These do not appear in the optimization operations, and the whole number N_{opt} of the beams to be optimized is related to the corresponding N_{beams} of the beams involved in the dynamic analysis, via the following:

$$N_{\mathrm{opt}} = N_{\mathrm{beams}} - 3 \tag{25}$$

Three cases have been considered for computational applications, corresponding to different polynomial degrees of the thickness series expansions in each optimized component beam. These computation examples can offer significant and illustrative results of the best thickness profile over each beam of the cantilever optimized truss structure, under the lowest vibration frequency constraint and allow us to assess the total structural mass reduction.

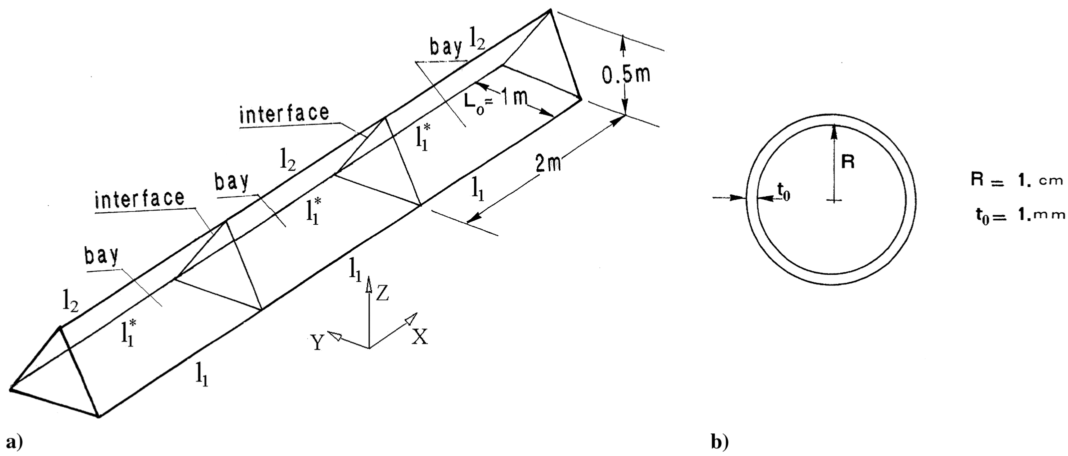

Fig. 2 Overall description of the truss structure: a) geometric shape of the periodic structure with three component bays, and b) cross section of the generic component beam.

V. Results

The minimum frequency occurs in the first symmetric flexural mode. That is, the out-of-plane deflections of the two lines l_1 and l_1^* are the same and nearly equal to the deflection of the line l_2 on the upper edge of the truss prismatic structure, as in Fig. 3 (similar to the modal shape in Fig. 4 of the aforementioned paper [9] with five bays). The corresponding minimum value of the nondimensional frequency parameter

$$\omega_d = \omega^2 \frac{\rho L_0^2}{E} \tag{26}$$

is 0.0084. The axial displacements along the longitudinal lines can be neglected.

1) First, the results obtained in the case with uniform thickness profile within each beam [$N_t = 1, i_t = 0$ in Eq. (7)], but with different values in the various beams, are shown in Table 1. The values on the aligned beams along the longitudinal lines l_1 and l_2 are listed (for symmetry reasons, the thickness profile on the line l_1^* is equal to the one on the line l_1); the ones on the crossbeams (the first belongs to the in-plane $l_1 l_1^*$ beam, and the other to the two out-of-plane beams with the same thickness value) at the right end of the first and second bay, respectively, are shown. As mentioned, the three crossbeams at the right free end of the third bay are not considered for the optimization operations.

The obtained relative weight $\mathcal{W}$ is about 0.87, that is, it is possible to save about 13% of the reference structure total weight.

2) The results obtained in the case with linear thickness behavior vs the axial coordinate X_l [$N_t = 2, i_t = 0, 1$ in Eq. (7)], are analyzed.

In Fig. 4, the linear thickness behavior on the three aligned beams along the longitudinal line l_1 is drawn vs this line's nondimensional axial coordinate

$$X_u = X_l + 2 \times (i_{\text{bay}} - 1) \qquad i_{\text{bay}} = 1, 2, \ldots, N_{\text{bays}} \tag{27}$$

where i_{bay} is the indicative number of each bay, growing from one at the left clamped edge to N_{bays} at the free end (in the case herein considered, $N_{\text{bays}} = 3$). This figure is divided into three parts : (a) the first, with $0 \le X_u = X_l \le 2$, corresponds to the first bay; (b) the second, with $2 \le X_u = X_l + 2 \le 4$, corresponds to the second bay; and (c) the third, with $4 \le X_u = X_l + 4 \le 6$, corresponds to the third bay.

As mentioned earlier, there is the possibility of discontinuity of the thickness value at the contact points between aligned beams of adjacent bays. The same behavior along line l_2 containing the three aligned beams, of (a) the first bay, (b) the second bay, and (c) the third bay, respectively, is sketched in Fig. 5. The thickness profile in this second case, with linear behavior vs the beam axial coordinate, is nearly uniform on the crossbeams, and its approximate values are as follows: in-plane/first bay = 0.831, out-of-plane/first bay = 0.856, in-plane/second bay = 0.822, and out-of-plane/second bay= 0.843. The relative weight $\mathcal{W}$ of the optimized structure in this second case is 0.869, which means that it is possible to save about 13.1% of the reference truss weight.

3) The third case refers to a quadratic thickness behavior vs the beam axial coordinate X_l [$N_t = 3, i_t = 0, 1, 2$ in Eq. (7)]. In Fig. 6, this optimized thickness behavior vs the nondimensional axial coordinate of the line l_1 is shown, considering separately the three aligned component beams of the first (a), second (b), and third bay (c), respectively.

The same behavior along the upper line l_2 is plotted in Figure 7. In this third case, the optimized thickness profile is nearly uniform in the in-plane crossbeams, and its approximate values are summarized as follows: in-plane/first bay = 0.829, and in-plane/second bay = 0.821.

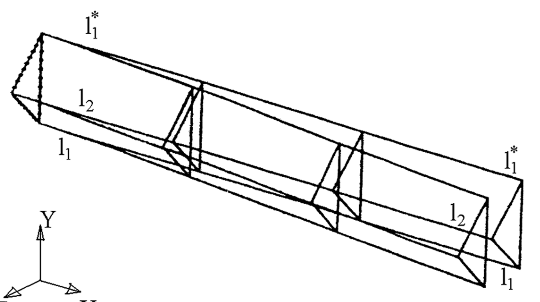

Fig. 3 Modal shape corresponding to the lowest natural frequency of the truss structure.

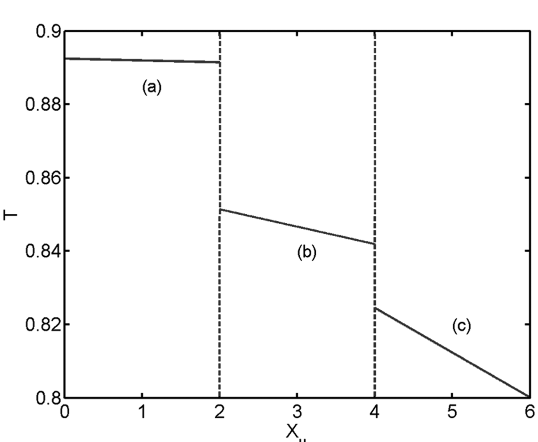

Fig. 4 Thickness optimized linear behavior along the longitudinal line l_1 axial coordinate.

Table 1 Thickness values in the various longitudinal and crossbeams of the optimized truss in the first considered case

	Line l_1	Line l_2	In-plane	Out-of-plane
First bay	0.909	1.067	0.824	0.861
Second bay	0.850	0.919	0.812	0.846
Third bay	0.800	0.806	——	——

The thickness quadratic optimized profile in the out-of-plane crossbeams vs their nondimensional normalized axial coordinate X_a, reformulated with respect their true length ($0 \leq X_a \leq 1$) at the right end of the first (a) and second (b) bay, respectively, is shown in Fig. 8. The relative weight $\mathcal{W}$ of the optimized structure in this third case is 0.864, which means that is possible to save about 13.6% of the reference truss weight.

VI. Discussion

In all the considered cases, the highest thickness value verifies on the first longitudinal beam aligned along the line l_2, whereas the minimum allowed value (corresponding to $T = 0.8$) occurs on the last longitudinal beams (at the free end), aligned along the line l_1 and l_1^* (as mentioned earlier, we have the same thickness throughout these lines).

In the second considered case, with a linear thickness profile, a very slight mass savings with respect to the first case, with uniform thickness within each beam, can be evinced.

In the third considered case, with a quadratic optimized profile, there is a more consistent weight reduction with respect to the previous second case, than the one obtained in the second case with respect to the first case. It could depend also on the out-of-plane crossbeams, where a quadratic symmetric optimized profile is obtained in the third case; this allows one to further reduce the structural weight with respect to the two other cases, where a uniform thickness is derived from optimization in these beams.

VII. Conclusions

This paper offers illustrative guidelines for applying a Ritz procedure, partially combined with FEM, for optimization of truss space structures, with a generic polynomial thickness optimized profile vs the axial coordinate of each beam. The simplified structural model of the satellite appendage can be considered as a first test case for the proposed computational method, which can be further improved and implemented for applications to more complex truss structures. The obtained results are significant enough to have an idea of the best thickness profile over each structural component beam of a cantilever truss under a vibration frequency constraint, and allows us to appreciate the mass saving.

Close examination of the results also reveals that the longitudinal beams play a fundamental role for the weight reduction, considering that the highest and the lowest thickness value occur on two of these beams. However, the out-of-plane crossbeams also affect the structural weight reduction. In fact, in the third considered case, with quadratic thickness behavior vs the axial coordinate of the structural beams, a higher mass savings can be obtained with respect to the two previous cases, where a uniform value is derived from the optimization operations in these crossbeams.

It is possible to emphasize that the improvement of weight savings obtained with the increasing degree of the polynomial describing functions of the thickness axial behavior is limited (from 13 to 13.6%). However, it is also necessary to point out that these obtained results are indicative of a particular simple structure. Application of

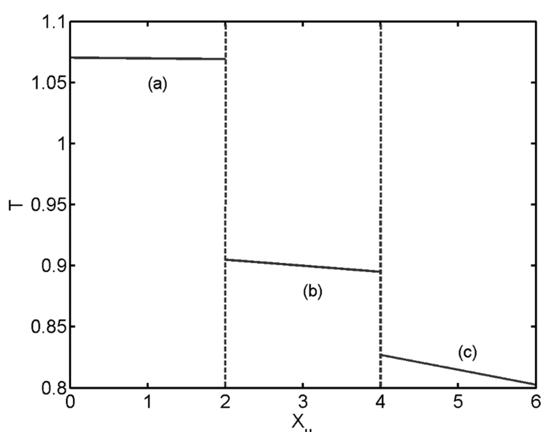

Fig. 5 Thickness optimized linear behavior along the longitudinal line l_2 axial coordinate.

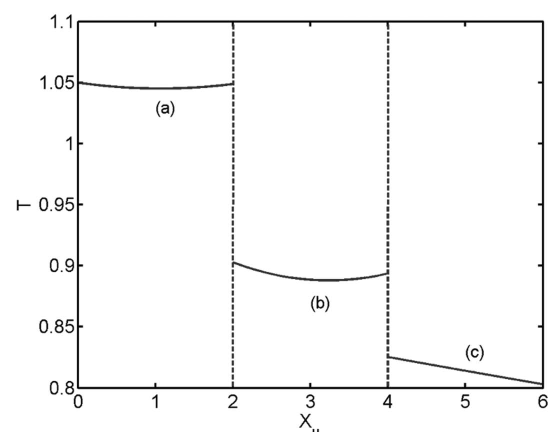

Fig. 7 Thickness optimized quadratic behavior along the longitudinal line l_2 axial coordinate.

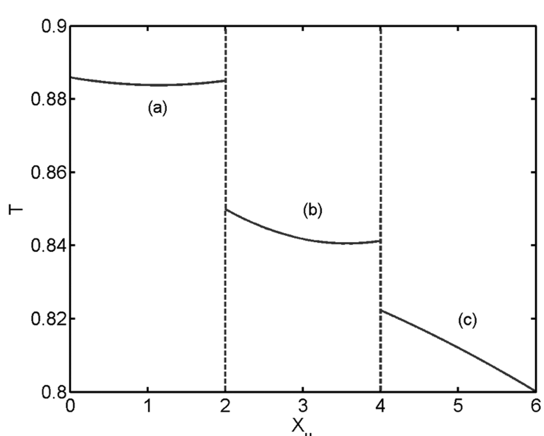

Fig. 6 Thickness optimized quadratic behavior along the longitudinal line l_1 axial coordinate.

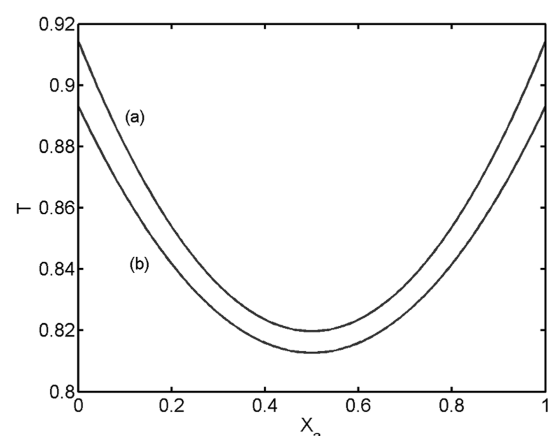

Fig. 8 Thickness optimized quadratic behavior along the out-of-plane crossbeams normalized axial coordinate.

this original procedure to dynamics and optimization of more complex and large truss structures could bring a much greater growth of weight reduction vs the increasing degree of the thickness behavior polynomial functions within each single component element.

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